The stability of elastico-viscous flow between rotating cylinders. Part 1

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Consideration is given to the flow of an idealized elastico-viscous liquid contained in the narrow channel between two concentric cylinders, the motion being due to the relative rotation of the cylinders. It is shown that the presence of elasticity in the liquid lowers the value of the critical Taylor number at which instability occurs. The secondary motion arising at the onset of instability has the usual cellular pattern, the cell length being decreased by the presence of elasticity in the liquid.

1. Introduction

Consideration has recently been given (Thomas & Walters 1963*a*) to the stability of flow of an elastico-viscous liquid in a narrow curved channel, in the case when the motion is due to a pressure gradient acting round the channel. The particular elastico-viscous liquid considered in that investigation was the liquid designated liquid B′ by Walters (1963), with equations of state

$$p_{ik} = -pg_{ik}+p'_{ik},\dagger \tag{1}$$

$$p'^{ik}(x,t) = 2\int_{-\infty}^{t} \Psi(t-t')\frac{\partial x^i}{\partial x'^m}\frac{\partial x^k}{\partial x'^r}e^{(1)mr}(x',t')\,dt', \tag{2}$$

where p_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} the metric tensor of a fixed co-ordinate system x^i, $e^{(1)}_{ik}$ the rate-of-strain tensor, and

$$\Psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau}e^{-(t-t')/\tau}\,d\tau. \tag{3}$$

In these equations, $N(\tau)$ is the relaxation spectrum (Walters 1960) and

$$x'^i = x'^i(x,t,t')$$

is the position at time t' of the element that is instantaneously at the point x^i at

† Covariant suffixes are written below, contravariant suffixes above, and the usual summation convention for repeated suffixes is assumed.

time t. The liquid designated liquid B by Oldroyd (1950) is a special case of liquid B′ obtained by writing

$$N(\tau) = \eta_0 \frac{\lambda_2}{\lambda_1}\,\delta(\tau) + \eta_0 \frac{\lambda_1-\lambda_2}{\lambda_1}\,\delta(\tau-\lambda_1),\dagger \tag{4}$$

where η_0 is the limiting viscosity at small rates of shear and λ_1 and λ_2 are the relaxation time and retardation time, respectively. The Newtonian liquid of constant viscosity η_0 is also a special case given by

$$N(\tau) = \eta_0 \delta(\tau). \tag{5}$$

Thomas & Walters (1963*a*) showed that, when the sides of the channel are stationary and the motion is due to a pressure gradient acting along the channel, the main effect of elasticity of type B′ is to *decrease* the value of the critical Reynolds number at which instability occurs. A more interesting problem from a practical standpoint is that in which the motion is due to the movement of the channel boundaries only; in particular, the motion between coaxial cylinders in relative rotation. Up to the present time, no work appears to have been done on the stability of elastico-viscous flow between rotating cylinders,‡ although the associated viscous-flow problem has been considered by a number of authors (see, for example, Taylor 1923; Chandrasekhar 1954).

In the present paper, we shall consider the stability of flow when liquid B′ is contained between coaxial cylinders in relative rotation. The method of solution is an extension of that used already by Chandrasekhar (1954). In order to use this method, it is found necessary to restrict the discussion to liquids with short memories—a description that can be applied to many real elastico-viscous liquids.

2. Steady-state solution

Cylindrical polar co-ordinates (r, θ, z) are chosen with the axis of the cylinders along the z-axis (which is drawn vertically upwards), and with the inner and outer cylinders having radii r_1 and r_2, respectively. It is supposed that the inner and outer cylinders rotate with angular velocities Ω_1 and Ω_2, respectively.

We consider the two-dimensional flow with velocity components§

$$v_{(r)} = 0, \quad v_{(\theta)} = V(r), \quad v_{(z)} = 0. \tag{6}$$

It can easily be shown (cf. Walters 1963; Thomas & Walters 1963*a*) that the corresponding stress distribution in the case of liquid B′ is

$$\left.\begin{aligned} p'_{(rr)} &= 0, \quad p'_{(\theta\theta)} = 2K_0\left\{r\frac{d}{dr}\left(\frac{V}{r}\right)\right\}^2, \quad p'_{(zz)} = 0, \\ p'_{(r\theta)} &= \eta_0 r \frac{d}{dr}\left(\frac{V}{r}\right), \quad p'_{(rz)} = 0, \quad p'_{(\theta z)} = 0, \end{aligned}\right\} \tag{7}$$

† δ denotes a Dirac delta-function, defined in such a way that

$$\delta(x) = 0 \ (x \neq 0); \int_{-\infty}^{\infty} \delta(x)\,dx = \int_0^{\infty} \delta(x)\,dx = 1.$$

‡ Graebel (1963) has considered the problem for a Bingham plastic solid. In this particular type of non-Newtonian flow, the presence of a yield value makes the laminar flow more stable.

§ Brackets placed round suffixes will be used throughout to denote *physical* components of tensors.

where $\eta_0 \left(= \int_0^\infty N(\tau)\, d\tau \right)$ is the limiting viscosity at small rates of shear and

$$K_0 = \int_0^\infty \tau N(\tau)\, d\tau.$$

If this distribution is now substituted into the stress equations of motion and the relevant boundary conditions are taken into account, we obtain the following expression for V,

$$V = Cr + D/r, \tag{8}$$

where

$$C = \frac{r_2^2\Omega_2 - r_1^2\Omega_1}{r_2^2 - r_1^2}, \quad D = \frac{r_1^2 r_2^2(\Omega_1 - \Omega_2)}{r_2^2 - r_1^2}. \tag{9}$$

The steady-state velocity distribution for liquid B′ is in fact identical with that for a Newtonian liquid of constant viscosity η_0.

3. The disturbance equations

We now consider the behaviour of the liquid when the steady state is disturbed slightly, confining the discussion to the case of neutral stability (cf. Taylor 1923; Chandrasekhar 1954). We assume a velocity distribution of the form

$$v_{(r)} = u, \quad v_{(\theta)} = V + v, \quad v_{(z)} = w, \tag{10}$$

where V is given by (8) and u, v, w are small quantities which are functions of r and z. In the following, we shall work to first order in u, v and w. The corresponding displacement functions (r', θ', z') are (Thomas & Walters 1963*a*)†

$$\left.\begin{aligned} r' &= r - (t-t')u, \\ \theta' &= \theta - (t-t')\left[\frac{V}{r} + \frac{v}{r}\right] + \frac{(t-t')^2}{2} u \frac{d}{dr}\left(\frac{V}{r}\right), \\ z' &= z - (t-t')w. \end{aligned}\right\} \tag{11}$$

In order to determine the contravariant rate-of-strain components $e^{(1)mr}(r', z', t')$ that appear in the equations of state (2), we write down the rate-of-strain components for the element at (r, θ, z) at time t, replace r, θ, z, t in these components by r', θ', z', t', and use (11). In this way, we obtain

$$\left.\begin{aligned} e^{(1)rr}(r',z',t') &= e^{(1)rr}(r,z,t,t') = \frac{\partial u(r,z)}{\partial r}, \\ e^{(1)\theta\theta}(r',z',t') &= e^{(1)\theta\theta}(r,z,t,t') = \frac{u(r,z)}{r^3}, \\ e^{(1)zz}(r',z',t') &= e^{(1)zz}(r,z,t,t') = \frac{\partial w(r,z)}{\partial z}, \\ e^{(1)rz}(r',z',t') &= e^{(1)rz}(r,z,t,t') = \frac{1}{2}\left[\frac{\partial w(r,z)}{\partial r} + \frac{\partial u(r,z)}{\partial z}\right], \\ e^{(1)z\theta}(r',z',t') &= e^{(1)z\theta}(r,z,t,t') = \frac{1}{2r}\frac{\partial v(u,z)}{\partial z}, \\ e^{(1)r\theta}(r',z',t') &= e^{(1)r\theta}(r,z,t,t') = \frac{1}{2}\left[\frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} + \frac{1}{r}\frac{\partial v(r,z)}{dr}\right. \\ &\quad \left. - \frac{v(r,z)}{r^2} - \frac{(t-t')u(r,z)}{r}\frac{d^2V}{dr^2} + \frac{2(t-t')u(r,z)}{r^2}\frac{dV}{dr} - \frac{2(t-t')u(r,z)}{r^3}V\right]. \end{aligned}\right\} \tag{12}$$

† The damping term $\exp[-(t-t')/\tau]$ in the equations of state ((2) and (3)) ensures that the effective range of $(t-t')$ is $(0, \xi)$ where ξ is some multiple of the highest relaxation time.

Equations (2), (11), (12) can now be used to determine the physical components of the partial stress tensor. After some reduction, we arrive at (cf. Thomas & Walters 1963*a*)

$$\left.\begin{aligned}
p'_{(rr)} &= 2\eta_0\frac{\partial u}{\partial r}, \quad p'_{(zz)} = 2\eta_0\frac{\partial w}{\partial z},\\
p'_{(\theta\theta)} &= \frac{2\eta_0 u}{r} + 2K_0\left\{\left(r\frac{dV}{dr}-V\right)\frac{\partial}{\partial r}\left(\frac{V}{r}+\frac{v}{r}\right)+\frac{d}{dr}\left(\frac{V}{r}\right)\left[r\frac{\partial v}{\partial r}-v\right]\right\}\\
&\quad + 2S_0\left\{2r^2\left[\frac{d}{dr}\left(\frac{V}{r}\right)\right]^2\frac{\partial u}{\partial r}+\left[r\frac{dV}{dr}-V\right]\left[-u\frac{d^2}{dr^2}\left(\frac{V}{r}\right)+\frac{d}{dr}\left(\frac{V}{r}\right)\frac{\partial u}{\partial r}\right]\right.\\
&\quad \left.+2\frac{d}{dr}\left(\frac{V}{r}\right)\left[-ur\frac{d^2V}{dr^2}+2u\frac{dV}{dr}-\frac{2uV}{r}\right]\right\},\\
p'_{(r\theta)} &= \eta_0\left[\frac{dV}{dr}-\frac{V}{r}\right]+\eta_0\left[\frac{\partial v}{\partial r}-\frac{v}{r}\right]+K_0\left\{3\frac{dV}{dr}\frac{\partial u}{\partial r}-\frac{3V}{r}\frac{\partial u}{\partial r}-u\frac{d^2V}{dr^2}\right.\\
&\qquad\qquad\left.+\frac{2u}{r}\frac{dV}{dr}-\frac{2uV}{r^2}\right\},\\
p'_{(z\theta)} &= \eta_0\frac{\partial v}{\partial z}+K_0\left\{\left(\frac{dV}{dr}-\frac{V}{r}\right)\frac{\partial w}{\partial r}+r\frac{d}{dr}\left(\frac{V}{r}\right)\left[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right]\right\},\\
p'_{(zr)} &= \eta_0\left[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right],
\end{aligned}\right\} \tag{13}$$

where $S_0 = \int_0^\infty \tau^2 N(\tau)\,d\tau$.

Substituting for V from (8) and writing $p''_{(ik)}$ for the additional stresses due to the disturbance, we obtain

$$\left.\begin{aligned}
p''_{(rr)} &= 2\eta_0\frac{\partial u}{\partial r}, \quad p''_{(zz)} = 2\eta_0\frac{\partial w}{\partial z},\\
p''_{(\theta\theta)} &= 2\eta_0\frac{u}{r}-\frac{8K_0D}{r}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{24D^2S_0}{r^4}\left[\frac{\partial u}{\partial r}+\frac{3u}{r}\right],\\
p''_{(r\theta)} &= \eta_0 r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)-\frac{6DK_0}{r^2}\left[\frac{\partial u}{\partial r}+\frac{u}{r}\right],\\
p''_{(rz)} &= \eta_0\left[\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right], \quad p''_{(z\theta)} = \eta_0\frac{\partial v}{\partial z}-\frac{2DK_0}{r^2}\left[2\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right].
\end{aligned}\right\} \tag{14}$$

The associated equations of motion are

$$\left.\begin{aligned}
-2\rho\left[C+\frac{D}{r^2}\right]v &= -\frac{\partial p^x}{\partial r}+\frac{\partial}{r\partial r}(rp''_{(rr)})+\frac{\partial p''_{(rz)}}{\partial z}-\frac{p''_{(\theta\theta)}}{r},\\
2\rho Cu &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2p''_{(r\theta)})+\frac{\partial}{\partial z}(p''_{(\theta z)}),\\
0 &= -\frac{\partial p^x}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(rp''_{(rz)})+\frac{\partial}{\partial z}(p''_{(zz)}),
\end{aligned}\right\} \tag{15}$$

where p^x is the additional pressure due to the disturbance, C and D are given by (9), and ρ is the density of the fluid. The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} = 0. \tag{16}$$

Substituting from (14) into (15) and using (16), we have

$$-2\rho\left[C+\frac{D}{r^2}\right]v = -\frac{\partial p^x}{\partial r}+\eta_0\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right] + \frac{8K_0 D}{r^2}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)-\frac{24D^2S_0}{r^5}\left(\frac{\partial u}{\partial r}+\frac{3u}{r}\right), \tag{17}$$

$$2\rho Cu = \eta_0\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right]-\frac{2K_0 D}{r^2}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial r^2}\right), \tag{18}$$

$$0 = -\frac{\partial p^x}{\partial z}+\eta_0\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right]. \tag{19}$$

To facilitate the analysis, we now make the usual assumption that the annular gap between the cylinders is small compared with the radii of the cylinders (cf. Taylor 1923; Chandrasekhar 1954). Substituting $r = r_1+dx$, where $d = r_2-r_1$, and $\alpha = (\Omega_2/\Omega_1)-1$ in equation (9) and assuming that d/r is small, we have

$$C \doteqdot \frac{r_1\alpha\Omega_1}{2d}, \quad D \doteqdot -\frac{r_1^3\alpha\Omega_1}{2d}. \tag{20}$$

Also
$$C+\frac{D}{r^2} \doteqdot \Omega_1[1+\alpha x]. \tag{21}$$

Under these conditions, equations (16)–(19) reduce to

$$\frac{1}{d}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0, \tag{22}$$

$$-2\rho\Omega_1[1+\alpha x]v = -\frac{1}{d}\frac{\partial p^x}{\partial x}+\eta_0\left[\frac{1}{d^2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right]-\frac{4K_0\Omega_1}{d^2}\frac{\alpha\partial v}{\partial x}-\frac{6S_0\Omega_1^2 r_1\alpha^2}{d^3}\frac{\partial u}{\partial x}, \tag{23}$$

$$\frac{\rho r_1\Omega_1\alpha u}{d} = \eta_0\left[\frac{1}{d^2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{dz^2}\right]+\frac{K_0 r_1\Omega_1\alpha}{d}\left(\frac{1}{d^2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{24}$$

$$0 = -\frac{\partial p^x}{\partial z}+\eta_0\left(\frac{1}{d^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right). \tag{25}$$

Making the usual assumption that the disturbance velocities are spatially periodic in the z-direction, it is possible to express equations (23)–(25) in non-dimensional form by using the following substitutions†

$$\left.\begin{aligned} u &= \epsilon R r_1\Omega_1\chi(x)\sin\lambda z,\\ v &= \epsilon r_1\Omega_1 v_1(x)\sin\lambda z\\ w &= Rr_1\Omega_1(d\chi/dx)\cos\lambda z,\\ p^x &= (\eta_0 r_1\Omega_1\epsilon R/d)\,p_1(x)\sin\lambda z, \quad R = d^2\Omega_1\rho/\eta_0, \quad \epsilon = \lambda d,\\ T &= -2\alpha r_1 R^2/d, \quad k = K_0/\rho d^2, \quad s = \eta_0 S_0/\rho^2 d^4. \end{aligned}\right\} \tag{26}$$

† For Oldroyd's liquid B, equation (4), $k = \eta_0(\lambda_1-\lambda_2)/(\rho d^2)$, $s = \eta_0^2\lambda_1(\lambda_1-\lambda_2)/(\rho^2 d^4)$; and $k = s = 0$ for the Newtonian liquid.

The parameter T is commonly referred to as the Taylor number. Substitution of (26) into (23)–(25) gives

$$-2[1+\alpha x]\,v_1 = -(dp_1/dx)+\nabla_1^2\chi-4k\alpha(dv_1/dx)+3\alpha sT(d\chi/dx), \tag{27}$$

$$\nabla_1^2 v_1 = \tfrac{1}{2}T[k\nabla_1^2\chi-\chi], \tag{28}$$

$$0 = -\epsilon^2 p_1+\left[\frac{d^3\chi}{dx^3}-\epsilon^2\frac{d\chi}{dx}\right], \tag{29}$$

where $\nabla_1^2 \equiv (d^2/dx^2)-\epsilon^2$. The equation of continuity is satisfied identically. Eliminating p_1 between (27) and (29) and writing $v_1 = v_0/2\epsilon^2$, we obtain

$$[1+\alpha x]\,v_0 = \nabla_1^4\chi+2k\alpha\frac{dv_0}{dx}-3s\epsilon^2\alpha T\frac{d\chi}{dx}, \tag{30}$$

$$\nabla_1^2 v_0 = -T\epsilon^2[\chi-k\nabla_1^2\chi]. \tag{31}$$

When $N(\tau) = \eta_0\delta(\tau)$, i.e. when $k = s = 0$, these equations reduce to those obtained by Chandrasekhar (1954) for the viscous case. Equations (30) and (31) determine a characteristic-value problem for the Taylor number T as a function of the wave number ϵ, and the lowest value of T for varying ϵ gives the critical conditions at which instability first sets in.

4. The solution of the disturbance equations

To facilitate the analysis, we restrict the discussion in the present paper to liquids with short memories, i.e. liquids with short relaxation times (cf. Walters 1962; Thomas & Walters 1963*b*). It is then possible to neglect terms involving $\int_0^\infty \tau^n N(\tau)\,d\tau$, $(n \geqslant 2)$, in comparison with those involving $\int_0^\infty \tau N(\tau)\,d\tau$ and $\int_0^\infty N(\tau)d\tau$. In the present paper, this approximation implies the neglect of terms involving s and terms of order k^2. Such an approximation would be justified, for example, in the case of the dilute polymer solutions investigated by Oldroyd, Strawbridge & Toms (1951). The simplified disturbance equations become

$$[1+\alpha x]\,v_0 = \nabla_1^4\chi+2k\alpha(dv_0/dx), \tag{32}$$

$$\nabla_1^2 v_0 = -T\epsilon^2[\chi-k\nabla_1^2\chi]. \tag{33}$$

Having modified the equations in this way, it is now possible to proceed, following the treatment given by Chandrasekhar (1954) in his discussion of the associated problem in viscous flow theory. We assume first a sine series expansion for v_0, which automatically satisfies the boundary conditions $v_0 = 0$ on $x = 0$ and $x = 1$. This expansion is then substituted into (32) and the expansion for χ deduced. Finally, these expressions are substituted into (33) and an infinite determinant is shown to vanish; this determinantal equation can be used to determine the minimum T for varying ϵ.

Following Chandrasekhar (1954) we write

$$v_0 = \sum_{p=1}^{\infty} A_p \sin \pi p x. \tag{34}$$

Substituting into equation (32), we obtain

$$\nabla_1^4 \chi = \sum_{p=1}^{\infty} A_p \left[\{1+\alpha x\} \sin \pi p x - 2\pi k \alpha p \cos \pi p x\right]. \tag{35}$$

The solution of this equation, which satisfies the boundary conditions

$$\chi = d\chi/dx = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = 1,$$

is

$$\chi = \sum_{p=1}^{\infty} \frac{A_p}{\pi^2 p^2 + \epsilon^2} \{C_1^{(p)} \cosh \epsilon x + D_1^{(p)} \sinh \epsilon x + C_2^{(p)} x \cosh \epsilon x + D_2^{(p)} x \sinh \epsilon x$$
$$+ (1+\alpha x) \sin \pi p x + 2\pi p \alpha \left[2 - k(p^2\pi^2 + \epsilon^2)\right] \cos \pi p x/(p^2\pi^2 + \epsilon^2)\}, \tag{36}$$

where $C_1^{(p)} = -2\pi p \alpha[2 - k(p^2\pi^2+\epsilon^2)]/(p^2\pi^2+\epsilon^2),$ (37)

$$D_1^{(p)} = (\pi p/\Delta)\left[\epsilon + \beta_p(\sinh \epsilon + \epsilon \cosh \epsilon) - \gamma_p \sinh \epsilon\right], \tag{38}$$

$$C_2^{(p)} = -(\pi p/\Delta)\left[\sinh^2 \epsilon + \epsilon(\sinh \epsilon + \epsilon \cosh \epsilon)\beta_p - \epsilon \sinh \epsilon \gamma_p\right], \tag{39}$$

$$D_2^{(p)} = (\pi p/\Delta)\left[(\cosh \epsilon \sinh \epsilon - \epsilon) + \epsilon^2 \sinh \epsilon \beta_p - (\epsilon \cosh \epsilon - \sinh \epsilon)\gamma_p\right], \tag{40}$$

$$\Delta = \sinh^2 \epsilon - \epsilon^2, \tag{41}$$

$$\beta_p = 2\alpha[2 - k(p^2\pi^2+\epsilon^2)]\{(-1)^{p+1} + \cosh \epsilon\}/(p^2\pi^2+\epsilon^2), \tag{42}$$

$$\gamma_p = (-1)^{p+1}(1+\alpha) + 2\alpha[2 - k(p^2\pi^2+\epsilon^2)]\epsilon \sinh \epsilon/(p^2\pi^2+\epsilon^2). \tag{43}$$

Substituting for v_0 and χ from (34) and (36) into (33) we have

$$\sum_{p=1}^{\infty} A_p [p^2\pi^2 + \epsilon^2] \sin \pi p x = T\epsilon^2 \sum_{p=1}^{\infty} \frac{A_p}{(p^2\pi^2+\epsilon^2)^2} \{\bar{C}_1^{(p)} \cosh \epsilon x + \bar{D}_1^{(p)} \sinh \epsilon x$$
$$+ C_2^{(p)} x \cosh \epsilon x + D_2^{(p)} x \sinh \epsilon x + \bar{F}^{(p)}[1+\alpha x] \sin \pi p x + \bar{E}^{(p)} \cos \pi p x\}, \tag{44}$$

where

$$\bar{C}_1^{(p)} = C_1^{(p)} - 2k\epsilon D_2^{(p)}, \tag{45}$$

$$\bar{D}_1^{(p)} = D_1^{(p)} - 2k\epsilon C_2^{(p)}, \tag{46}$$

$$\bar{F}^{(p)} = 1 + k(\pi^2 p^2 + \epsilon^2), \tag{47}$$

$$\bar{E}^{(p)} = 4\pi p \alpha/(p^2\pi^2+\epsilon^2). \tag{48}$$

Multiplying equation (44) by $\sin \pi q x$ and integrating from $x = 0$ to $x = 1$, we obtain

$$\sum_{p=1}^{\infty} \left(\frac{q\pi}{q^2\pi^2+\epsilon^2} \left\{ [1 + (-1)^{q+1} \cosh \epsilon] \bar{C}_1^{(p)} + [(-1)^{q+1} \sinh \epsilon] \bar{D}_1^{(p)} \right.\right.$$
$$+ \left[(-1)^{q+1} \cosh \epsilon + \frac{2(-1)^q \epsilon \sinh \epsilon}{q^2\pi^2+\epsilon^2}\right] C_2^{(p)}$$
$$\left. + \left[(-1)^{q+1} \sinh \epsilon - \frac{2\epsilon\,[1 + (-1)^{q+1} \cosh \epsilon]}{q^2\pi^2+\epsilon^2}\right] D_2^{(p)} \right\}$$
$$\left. + X_{pq} - \tfrac{1}{2}(\pi^2 p^2 + \epsilon^2)^3 \frac{\delta_{pq}}{\epsilon^2 T} \right) \Lambda_p = 0, \tag{49}$$

where $\Lambda_p = A_p/(\pi^2 p^2 + \epsilon^2)^2$ and

$$\left.\begin{aligned} X_{pq} &= (\tfrac{1}{2} + \tfrac{1}{4}\alpha)\bar{F}^{(p)}, \quad \text{when} \quad p = q, \\ X_{pq} &= 0, \quad \text{when} \quad p \neq q \quad \text{and} \quad p+q \quad \text{is even}, \\ X_{pq} &= -\frac{4pq\alpha \bar{F}^{(p)}}{\pi^2(q^2-p^2)^2} + \frac{2q\bar{E}^{(p)}}{\pi(q^2-p^2)}, \quad \text{when} \quad p \neq q \quad \text{and} \quad p+q \quad \text{is odd.} \end{aligned}\right\} \tag{50}$$

Equations (49) represents a system of linear homogeneous equations for the Λ_p's. The condition that not all the Λ_p's vanish is that the determinant of the system should vanish, i.e.

$$\left\| \frac{q\pi}{q^2\pi^2+\epsilon^2}\left\{[1+(-1)^{q+1}\cosh\epsilon]\,\bar{C}_1^{(p)}+[(-1)^{q+1}\sinh\epsilon]\,\bar{D}_1^{(p)} + \left[(-1)^{q+1}\cosh\epsilon+\frac{2(-1)^q\epsilon\sinh\epsilon}{q^2\pi^2+\epsilon^2}\right]C_2^{(p)} + \left[(-1)^{q+1}\sinh\epsilon-\frac{2\epsilon\,[1+(-1)^{q+1}\cosh\epsilon]}{q^2\pi^2+\epsilon^2}\right]D_2^{(p)}\right\}+X_{pq}-\tfrac{1}{2}(p^2\pi^2+\epsilon^2)^3\frac{\delta_{pq}}{\epsilon^2T}\right\| = 0. \tag{51}$$

Equation (51) is an equation in the two variables T and ϵ. The critical Taylor number T_c at which the laminar flow pattern breaks down is determined by calculating the minimum T for varying ϵ. The value of ϵ corresponding to T_c will be denoted by ϵ_c.

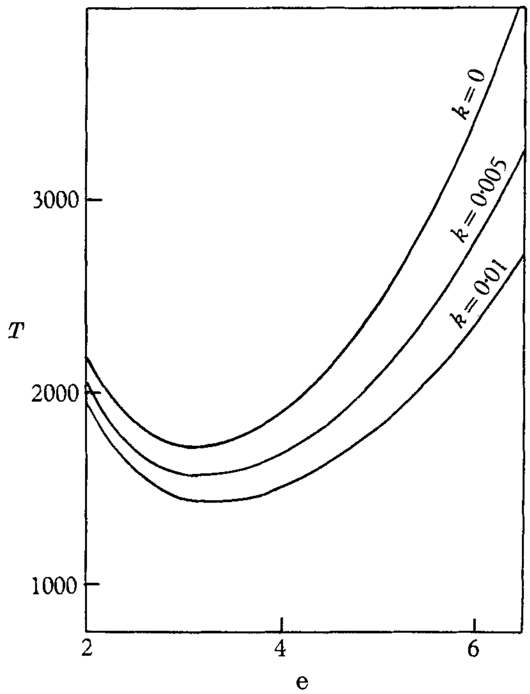

FIGURE 1. Graphs of T against ϵ for various values of k; $\alpha = 0$.

5. Results

A second-order determinant was used in the calculation of the critical Taylor number T_c. The error involved was estimated by the present authors to be less than 1 % (cf. Chandrasekhar 1954). The graphs and table indicate that the presence of a very small amount of elasticity in the fluid lowers the critical Taylor number by as much as 20 %, so that the second-order approximation is more than adequate to illustrate the dependence of T_c on k, over the permissible

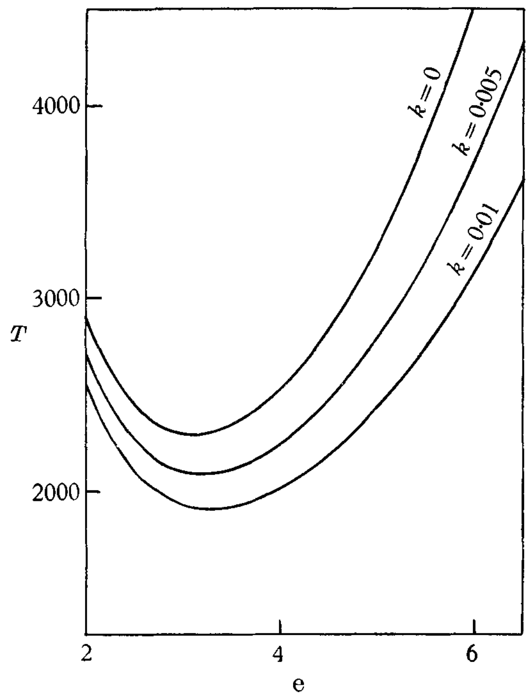

FIGURE 2. Graphs of T against ϵ for various values of k; $\alpha = -0{\cdot}5$.

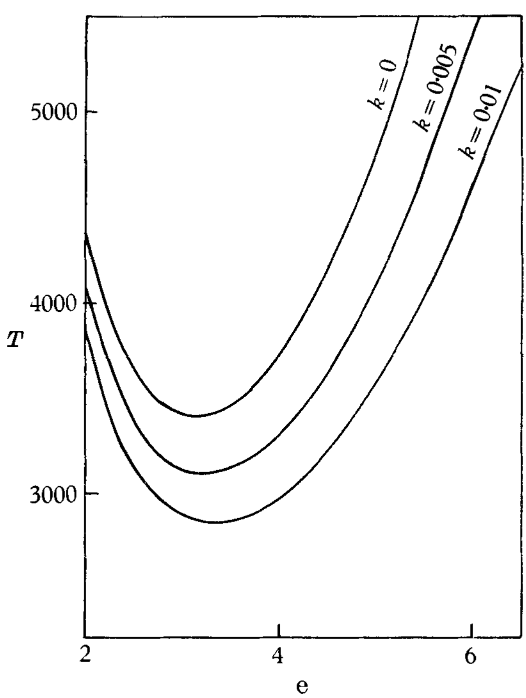

FIGURE 3. Graphs of T against ϵ for various values of k; $\alpha = -1{\cdot}0$.

range of k. Figures 1, 2 and 3 illustrate the dependence of T_c on k for various values of α. It is seen that a slight increase in k leads to a spectacular decrease in T_c. The dependence of T_c on k is further illustrated in table 1 and figure 4. Also, the values of ϵ associated with the minimum Taylor numbers are greater for the elastico-viscous liquids than for the Newtonian liquid (table 1). Thus,

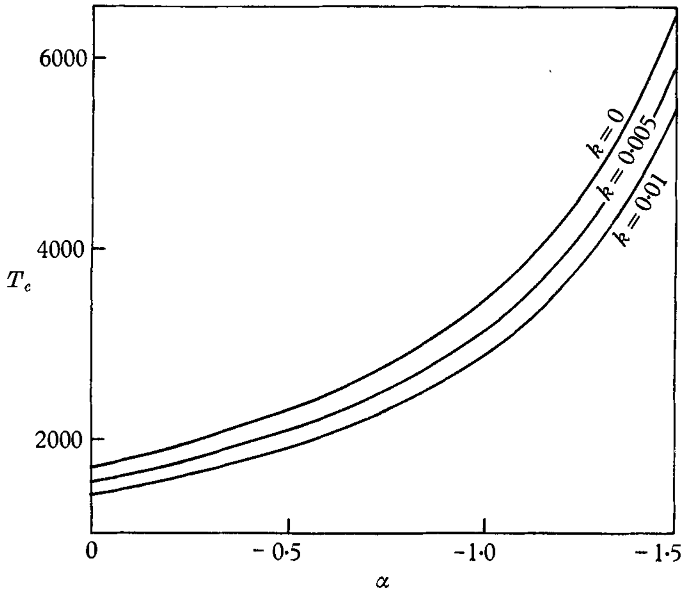

FIGURE 4. Graphs of T_c against α for various values of k.

	(i) $k = 0$		(ii) $k = 0{\cdot}005$		(iii) $k = 0{\cdot}01$	
α	T_c	ϵ_c	T_c	ϵ_c	T_c	ϵ_c
0	1715	3·12	1560	3·22	1427	3·31
−0·25	1960	3·12	1783	3·22	1631	3·31
−0·5	2285	3·12	2078	3·22	1902	3·31
−0·75	2736	3·12	2490	3·22	2279	3·31
−1·00	3404	3·12	3098	3·22	2840	3·32
−1·25	4478	3·15	4079	3·25	3754	3·35
−1·5	6431	3·20	5874	3·31	5467	3·41

TABLE 1. Values of T_c and ϵ_c for various values of k and α.

the cells which arise at the onset of instability (Taylor 1923) are decreased in length by the presence of elasticity in the liquid.

These results are in agreement with the findings of Thomas & Walters (1963a) in their consideration of the associated problem when the motion is due to a pressure gradient acting along the channel.

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